Selection Procedures for Log-Location-Scale Distributions

MAT 752 – Fall 2016 – Guest Lecture

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October 24, 2016

Applications of Lognormal Distribution

- Lifetime Data (Reliability and Survival Analysis)
  - failure times (semiconductor devices, shock absorbers,etc.)
  - survival times of patients with certain types of cancer
- Emissions Data (Quality Control)
- Characteristics from other fields
  - stock/portfolio price returns (finance)
  - income, price change, household size(economics)
  - insurance claim payments (actuarial science)
  - particle size (geology)

Definitions and Notations

$X \sim \Lambda(\mu, \sigma^2)$ (log-normal (μ, σ^2)), if $Y = \ln X \sim N(\mu, \sigma^2)$.

Some important quantities associated with X:

- $EX = \exp(\mu + \frac{1}{2}\sigma^2)$
- $E(X^m) = \exp(m\mu + \frac{m^2}{2}\sigma^2)$
- $\text{Median}(X) = \exp(\mu)$
- $\text{Mode}(X) = \exp(\mu - \sigma^2)$
- α - quantile of X, $\tau_\alpha = \exp(\mu + \Phi^{-1}(\alpha)\sigma)$

Parameter of interest:

$\theta = \theta(a, b) = \exp(a\mu + b\sigma^2)$ or
$\tau = \tau(a, b) = \exp(a\mu + b\sigma)$

<u>Equal σ_i^2</u> $\Rightarrow$ Normal Selection Procedures

Known Case: Procedure from Bechhofer(1954) seminal paper
Unknown Case: Two-Stage Procedures
Dudewicz & Dalal (1975) or Rinott (1978)

Unequal σ_i^2:

Let $\beta = \ln\theta$ and $\eta = \ln\tau$.

Define Ω_0 (Indifference Zone) based on β or τ using distance measure $d(u, v) = u - v$.

Unequal and Known σ_i^2 Case:

[John and Chen(2006a)]

Procedure $\mathcal{R}.1$

Observe X_{ij}, $j = 1, ..n_i$ from Π_i.

Let $\bar{Y}_i = (\sum_{j=1}^{n_i} \ln X_{ij})/n_i$ and define $\widehat{\beta}_i$ and $\widehat{\eta}_i$ with $\widehat{\mu}_i = \bar{Y}_i$.

Select the population with the largest $\widehat{\beta}_i$ (or $\widehat{\eta}_i$).

How to set sample size(s)?

Theorem 1

If $\dfrac{\sigma_1^2}{n_1} = ... = \dfrac{\sigma_k^2}{n_k} = \dfrac{(\delta *)^2}{a^2 c^2}$ *and* c *is chosen to satisfy the integral equation,*

$$\int_{-\infty}^{\infty} \left[\Phi\left(y + c\right)\right]^{k-1} \phi(y) \ dy = P^* \tag{1}$$

then $P_{\Omega_1}(CS \mid \mathcal{R}.1) \geq P^*$.

c is calculated from equation 1

sample sizes are given by $n_i = \dfrac{a^2 c^2 \sigma_i^2}{(\delta *)^2}$.

Unknown and Unequal σ_i^2

John & Chen (2006): IEEE Transactions on Reliability
Two-Stage Asymptotic Procedure for $\theta_i = \exp(a\mu_i + b\sigma_i^2)$:

Procedure $\mathcal{R}.2$

Stage 1: *Observe* X_{ij}, $j = 1, \ldots, N_0$ *from* Π_i.
Let $S_{0,i}^2$ = *UMVUE of* σ_i^2 *from* $Y_{ij} = \ln X_{ij}$ *and let*

$$N_i = max \left\{ N_0, \left\lceil \frac{a^2\, l^2\, S_{i,0}^2}{(\delta^*)^2} \right\rceil, \left\lceil \frac{2b^2\, l^2\, S_{i,0}^4}{(\delta^*)^2} \right\rceil \right\}, \qquad (2)$$

$l = l(k, N_0, P^*)$ *is to be defined and* $\lceil x \rceil = \mathrm{ceil}(x)$.

Stage 2: *Observe additional* X_{ij}, $j = N_0 + 1, \ldots, N_i$.
Calculate $\bar{Y}_i$ *and* S_i^2 *from the* **total** *samples.*
Let $\widehat{\beta}_i = \widehat{\beta}_i(a, b) = a\bar{Y}_i + bS_i^2$
Select the population that produced $\widehat{\beta}_{[k]}$.

Theorem 2

Define

$$h(I,x,y) = \begin{cases} \sqrt{x+x^2+y+y^2}, & a \neq 0,\ b \neq 0 \\ \sqrt{x+y}, & a \neq 0,\ b = 0 \\ \sqrt{x^2+y^2}, & a = 0,\ b \neq 0 \end{cases}$$

$$f_p(t) = \frac{p^p}{\Gamma(p)} \frac{e^{-p/t}}{t^{p+1}}, \ \textit{with } p = \frac{N_0 - 1}{2}.$$

If I is chosen to satisfy the integral equation

$$\int_0^\infty \left(\int_0^\infty \Phi\left(\frac{I}{h(x,y|a,b)} \right) f_p(x)dx \right)^{k-1} f_p(y)dy = P^*, \qquad (3)$$

then asymptotically $P_{\Omega_1}(CS \mid \mathcal{R}.2) \geq P^*$.

Theorem 3

Let

$$I_1(y, u_1, u_2) = \int_{u_1}^{u_2} h(l, x, y) f_p(x) dx$$

$$I_2(u_1, u_2, v_1, v_2) = \int_{v_1}^{v_2} \left(I_1(y, u_1, u_2)\right)^{k-1} f_p(y) dy.$$

If $A_1 < \frac{p}{p+1}$ *and* $B_1 < \frac{p}{p+1}$, *then*

$$|I_2(0, \infty, 0, \infty) - I_2(A_1, A_2, B_1, B_2)|$$

$$\leq B_1 f_p(B_1) + \frac{p^{p-1}}{\Gamma(p) B_2^p} + (k-1)\left(A_1 f_p(A_1) + \frac{p^{p-1}}{\Gamma(p) A_2^p}\right).$$

For example $k = 3$ and $N_0 = 5$ case:
Integration on $(0.1, 500)$ is enough for $\mathrm{error} \leq 0.00003$.

Example 1: Reliability

Time to electromigration failure of aluminum conductors in microcircuits (Nelson & Doganaksoy *Recent Advances in Life-Testing & Reliability* - 1995):

4 independent processes - With 95% guarantee find the process with the largest mean time to electromigration failure which is at least 20% more than that of the inferior conductors
(So $k = 4$, $\delta^* = \ln 1.2 \cong 0.1823$ and $P^* = 0.95$):

Set $N_0 = 15 \Rightarrow l = 4.3296$

From Stage 1 samples:

$$S^2_{0,1} = 0.0480 \Rightarrow N_1 = 28$$
$$S^2_{0,2} = 0.0425 \Rightarrow N_2 = 24$$
$$S^2_{0,3} = 0.0698 \Rightarrow N_3 = 40$$
$$S^2_{0,4} = 0.1110 \Rightarrow N_4 = 63$$

From Total samples $\hat{\beta}$'s are observed:
$\hat{\beta}_1 = 1.945$, $\hat{\beta}_2 = 1.957$, $\hat{\beta}_3 = 2.009$, and $\hat{\beta}_4 = 2.08$.

Example 2: Quality Control

CO-emissions (McDonald, Vance, and Gibbons - *Recent Advances in Life-Testing & Reliability* - 1995):

7 models- With 95% guarantee find the model with the smallest mean *CO* emissions which is at most half of the other models
(So $k = 7$, $\delta^* = \ln 2 \cong 0.69315$ and $P^* = 0.95$):
Set $N_0 = 15 \Rightarrow l = 4.9973$

From Stage 1 samples:

$S^2_{0,1} = 0.1239 \Rightarrow N_1 = 15$	$S^2_{0,2} = 0.06295 \Rightarrow N_2 = 15$
$S^2_{0,3} = 0.2452 \Rightarrow N_3 = 15$	$S^2_{0,4} = 0.41270 \Rightarrow N_4 = 22$
$S^2_{0,5} = 1.0086 \Rightarrow N_5 = 53$	$S^2_{0,6} = 0.76291 \Rightarrow N_6 = 40$
$S^2_{0,7} = 0.8947 \Rightarrow N_4 = 47$	

From Total samples $\hat{\beta}$'s are observed:
$\hat{\beta}_1 = 0.0933$, $\hat{\beta}_2 = 0.8405$, $\hat{\beta}_3 = 0.50821$, $\hat{\beta}_4 = 0.92574$, $\hat{\beta}_5 = 0.4281$, $\hat{\beta}_6 = 0.6189$, and $\hat{\beta}_7 = 1.0001$.

Two-Stage Asymptotic Procedure For $\tau_i(a, b)$:

Recall $\tau_i(a, b) = \exp(a\mu_i + b\sigma_i)$.
Denote $\delta_1^* = \delta^* / \sqrt{a^2 + (b^2/2)}$.

Procedure $\mathcal{R}.3$

Stage 1: Calculate $\bar{Y}_{0,i}$ and $S_{0,i}^2$ from Π_i and define

$$N_i = max\left\{N_0, \left\lceil \frac{l^2 S_{0,i}^2}{(\delta_1^*)^2} \right\rceil \right\}$$

where l is given by (3) with $a = 1$ and $b = 0$.
Stage 2: Take additional $N_i - N_0$ observations from Π_i.
From the **total** *samples, calculate $\widehat{\eta}_i = a\bar{Y}_i + b\sqrt{S_i^2}$.*
Select the population with $\widehat{\eta}_{[k]}$.

Theorem 4

Asymptotically $P_{\Omega_1}(CS \mid \mathcal{R}.3) \geq P^*$.

Comparison to Sobel's Nonparametric Procedure

Let $\mathcal{R}_{\mathcal{S}}$ denote the nonparametric procedure (single stage with fixed sample sizes) of [Sobel(1967)]
Ω_1 is defined for log-normal α-quantile with fixed α as:

$$\begin{aligned} \mathcal{R}_{\mathcal{S}}: &\quad \mu_{(k-1)} + \Phi^{-1}(\alpha + \epsilon_2^*)\sigma_{(k-1)} \leq \mu_{(k)} + \Phi^{-1}(\alpha - \epsilon_1^*)\sigma_{(k)} \\ \mathcal{R}.3: &\quad \mu_{(k-1)} + \Phi^{-1}(\alpha)\sigma_{(k-1)} \leq \mu_{(k)} + \Phi^{-1}(\alpha)\sigma_{(k)} - \delta^* \end{aligned}$$

We set

$$\Phi^{-1}(\alpha+\epsilon_2^*)\sigma_{(k-1)}-\Phi^{-1}(\alpha-\epsilon_1^*)\sigma_{(k)} = \Phi^{-1}(\alpha)\sigma_{(k-1)}-\Phi^{-1}(\alpha)\sigma_{(k)}+\delta^*$$

Sobel(1967) Formulation 2:

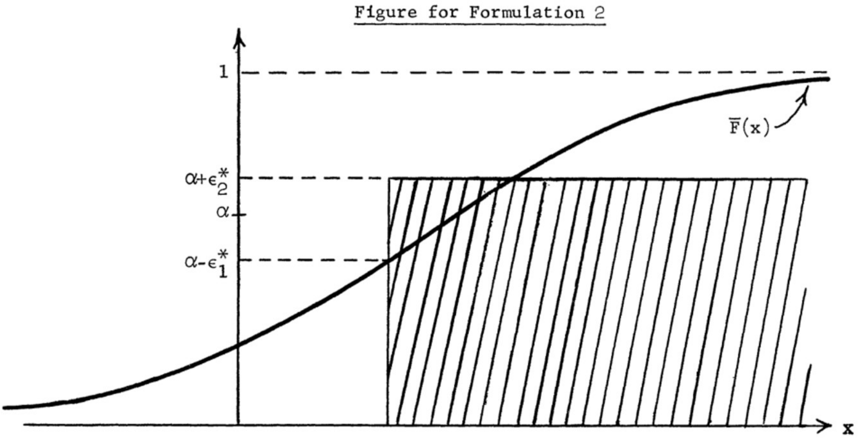

FIGURE for Formulation 2. To be in the preference zone (i.e., in the complement of the indifference zone) each $F_{[i]}(x)$ $(i = 1, 2, \cdots, k - t)$ must avoid the shaded area, i.e., $\underline{F}(x)$ must avoid the shaded area.

Comparison of John & Chen (2006) to Sobel(1967)

Consider $\alpha = 0.99$, $\epsilon_1^* = \epsilon_2^* = 0.005$, $k = 3$, and $P^* = 0.90$.

$\mathcal{R}_{\mathcal{S}}$ requires a sample size of $n = 1158$.

The estimated EN for Procedure $\mathcal{R}.3$ with $N_0 = 5$ are smaller than 417 (some even single digits), except for two cases; and even in those cases they are smaller than 971.

When $N_0 = 15$, the estimated EN are even smaller: EN are even ≤ 323, except for two cases; and overall EN are ≤ 740.

Overall the estimated EN for procedure $\mathcal{R}.3$ are overall much smaller than that for $\mathcal{R}_{\mathcal{S}}$. Exceptions are when δ^* is small, or σ is large.

Note: When $\alpha = 0.5$ the comparison of Procedure $\mathcal{R}.3$ with $\mathcal{R}_{\mathcal{S}}$ is indeed a comparison of Rinott's two-stage procedure with Sobel's nonparametric procedure. We also note that the table for the $\alpha = 0.5$ case provided in [Sobel(1967)] was used to check for the accuracy of our computations.

Followup Work

An enhanced lognormal selection procedure

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Discrete Event Dynamic Systems, 2011, V 21 (2), pp 205-218

Abstract:

John and Chen (IEEE Trans Reliab 55(1):135âĂŞ148, 2006) propose an exact two-stage solution based on the Least Favorable Configuration (LFC) to the ranking and selection problem of determining the best system from k lognormal populations. Lognormal density is commonly used to model certain lifetimes in reliability and survival analysis. It is known that selection procedures that are developed based on the LFC are conservative and become inefficient when the number of systems is large. We propose to take into account the differences of sample values within the parameter of interest when determining sample sizes, which can significantly increase the efficiency of the selection procedures. We also sequentialize the selection procedure and provide a procedure for estimating the constant needed to apply the solution. An experimental performance evaluation demonstrates the validity and efficiency of the enhanced lognormal selection procedure.

Followup Work: Sequential LN Selection Procedure

The proposed procedure is as follows.

The Sequential Lognormal Selection algorithm:

1. Initialize the set I to include all k lognormal populations. Simulate n_0 samples for each system $i \in I$. Set the iteration number $j = 0$, and $N_{1,j} = N_{2,j} = \ldots = N_{k,j} = n_0$, where $N_{i,j}$ is the sample size allocated for system i in the j^{th} iteration.
2. Compute the needed incremental number of samples for system i in the $(j+1)^{th}$ iteration $\delta_{i,j+1} = \lceil (\max((hS_{i,j}/\hat{d}_{i,j})^2, (hS_{i,j}^2/\hat{d}_{i,j})^2/2) - N_{i,j})^+/2 \rceil$, where $\hat{d}_{i,j} = \max(d_a^*, L(\hat{\beta}_{b,j}) - \hat{\beta}_{i,j})$ and $(x)^+ = \max(0, x)$.
3. If $\delta_{i,j+1} = 0$ and $i \neq b$ (where $\hat{\beta}_{b,j} = \max_{i \in I} \hat{\beta}_{i,j}$), remove system i from the subset I.
4. If there is only one element in the subset I, go to step 6.
5. Simulate $\delta_{i,j+1}$ additional samples for each system $i \in I$ in the $(j+1)^{th}$ iteration. Set $j = j + 1$ and go to step 2.
6. Return the values b and $\hat{\beta}_b$, where $\hat{\beta}_b = \max_{1 \leq i \leq k} \hat{\beta}_i$.

We can reduce the number of iterations with a larger incremental sample size $\delta_{i,j}$ for system i in the j^{th} iteration, but we run the risk of allocating more samples than necessary to non-promising systems. We propose to compute $\delta_{i,j}$ dynamically with all

Followup Work: Comparison of Expected Sample Sizes

Discrete Event Dyn Syst (2011) 21:205–218 215

Table 2 The observed P(CS) with $P^* = 0.90$ and $k = 3$

n_0	JC				SLS				
	$\widehat{EN_1}$	$\widehat{EN_2}$	$\widehat{EN_3}$	$\widehat{P(CS)}$	$\widehat{EN_1}$	$\widehat{EN_2}$	$\widehat{EN_3}$	$\widehat{P(CS)}$	Iter
5	620	735	907	0.949	394	529	752	0.985	10
15	173	213	276	0.930	117	152	250	0.961	9
25	146	180	229	0.929	104	134	210	0.933	8
35	134	167	213	0.935	101	123	196	0.933	8
45	126	162	205	0.945	99	120	190	0.920	8

Under the LFC, the reduction of sample sizes is mainly the result of sequentializing the procedure because $\hat{d}_{lk} \approx d_a^*$. The sequentialized procedure reduces the frequency of over-allocating samples.

Notations:

Let $\Pi_i, i = 1, \ldots, k$ be k independent populations with distribution function $F_i = F(x : \mu_i, \sigma_i)$.

Location parameter $\mu_i \in \mathbb{R}$
Scale parameter $\sigma_i \in \mathbb{R}_+$.

For a known distribution function G,

$$F_i(t) = G\left(\frac{t - \mu_i}{\sigma_i}\right)$$

Sub-class of Location-Scale Distributions

Three Assumptions:

- Assumption 1: F_i is absolutely continuous with density f_i.
- Assumption 2: $\frac{d^2}{dt^2}(\log g(t))$ is continuous for $t \in \mathbb{R}$.
- Assumption 3: F_i satisfies the conditions for information inequality (in particular, $g > 0$ on $\mathbb{R}$)

$g > 0$ on $\mathbb{R} \Rightarrow \{f_i > 0\}$ does not depend on μ_i

Distributions that satisfy the assumptions:
Normal, Extreme Value, Logistic

Distributions that fail the assumptions: *Uniform, Exponential*

If $Y = \log X$ has a location-scale distribution,
X has a log-location-scale distribution.
Examples: *Log-Normal, Weibull, Log-Logistic*

Type-II Censored Data from Log-Location-Scale

X an observation from Log-Location-Scale

$Y = \ln X$ corresponding observation from Location-Scale

$Z = \frac{Y-\mu}{\sigma}$ corresponding ("standardized") observation from the generating distribution

$Y_{j:n}$ = the j-th ordered observation

Smallest r_1 and the largest r_2 are censored.

Let $p_1 = \frac{r_1}{n}$ and $p_2 = \frac{r_2}{n}$ be the censoring proportions.

Estimating μ:

Several BAN estimates of μ from censored samples:
MLE, BLUE, ABLUE, etc.

Doubly Type-II Censored Sample: Variance of $\widehat{\mu}_i$

$$B_0(t) = -\frac{d^2}{dt^2}\log g(t).$$
$$B_1(t) = -\frac{d^2}{dt^2}\log G(t).$$
$$B_2(t) = -\frac{d^2}{dt^2}\log(1 - G(t)).$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial\mu^2} &= -\frac{1}{\sigma^2}\left\{\sum_{l=r_1+1}^{n-r_2} B_0(z_{l:n}) + r_1 B_1(z_{r_1+1:n}) + r_2 B_2(z_{n-r_2:n})\right\} \\ &= -\frac{n}{\sigma^2}\left\{\frac{1}{n}\sum_{l=r_1+1}^{n-r_2} B_0(z_{l:n}) + p_1 B_1(z_{r_1+1:n}) + p_2 B_2(z_{n-r_2:n})\right\}\end{aligned}$$

$$B(p_1, p_2) = \int_{q_1}^{q_2} B_0(t)\, g(t)\, dt \; + p_1 B_1(q_1) + p_2 B_2(q_2)$$

The asymptotic variance of $\widehat{\mu}$

Let $p_1 = \frac{r_1}{n}$, and $p_2 = \frac{r_2}{n}$ be fixed and let $n \to \infty$

Let $q_1 = G^{-1}(p_1)$ and $q_2 = G^{-1}(1 - p_2)$.

$z_{r_1+1:n} \to G^{-1}(p_1)$, $z_{n-r_2:n} \to G^{-1}(1 - p_2)$ and

$$\lim_{n\to\infty} E\left(\frac{1}{n}\sum_{l=r_1+1}^{n-r_2} B_0(z_{l:n})\right) = \int_{q_1}^{q_2} B_0(t)\, g(t)\, dt$$

The asymptotic variance of $\widehat{\mu}$ is $\quad AV(\widehat{\mu}) \approx \dfrac{\sigma^2}{n\, B(p_1, p_2)}$

This generalizes [Harter and Moore(1966)]

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The elements of the information matrix (multiplied by σ^2/n) may be written as

$$\lim_{n\to\infty} -\frac{\sigma^2}{n} E\left[\frac{\partial^2 L}{\partial \mu^2}\right] = p + f(\hat{z}_1)\left[\hat{z}_1 + \frac{f(\hat{z}_1)}{q_1}\right] - f(\hat{z}_2)\left[\hat{z}_2 - \frac{f(\hat{z}_2)}{q_2}\right] = v_{11}, \tag{3·7}$$

$$\lim_{n\to\infty} -\frac{\sigma^2}{n} E\left[\frac{\partial^2 L}{\partial \mu\, \partial \sigma}\right] = f(\hat{z}_1) - f(\hat{z}_2) + \hat{z}_1 f(\hat{z}_1)\left[\hat{z}_1 + \frac{f(\hat{z}_1)}{q_1}\right] - \hat{z}_2 f(\hat{z}_2)\left[\hat{z}_2 - \frac{f(\hat{z}_2)}{q_2}\right] = v_{12}, \tag{3·8}$$

$$\lim_{n-\infty} -\frac{\sigma^2}{n} E\left[\frac{\partial^2 L}{\partial \sigma^2}\right] = 2p + \hat{z}_1 f(\hat{z}_1) - \hat{z}_2 f(\hat{z}_2) + \hat{z}_1^2 f(z_i)\left[\hat{z}_1 + \frac{f(\hat{z}_1)}{q_1}\right] - \hat{z}_2 f(\hat{z}_2)\left[\hat{z}_2^2 - \frac{f(\hat{z}_2)}{q_2}\right] = v_{22}. \tag{3·9}$$

The asymptotic variance-covariance matrix for the estimators $\hat{\mu}$ and $\hat{\sigma}$ is then $\sigma^2[\sigma_{ij}]/n$, where $[\sigma_{ij}] = [v_{ij}]^{-1}$. If one drops the terms involving $\hat{z}_2$ from the equations (3·7)–(3·9) the results agree with those given by Gupta for the case of single censoring.

The generalization is:

from estimation of normal mean [Harter and Moore(1966)]

to estimation of the *location parameter* from location-scale distributions [John and Chen(2008)]

Assessment of Approximation

Table 1
For Type II proportion-p_2-right-censored Normal samples, B_E = (exact) values of the left side in (19), along with the approximate values $B = B(0, p_2)$. See Sec. 4.4 for the discussion

n	p_2	B_E	B
10	0.9	0.31401	0.35913
	0.8	0.49698	0.53360
	0.7	0.62602	0.65503
	0.6	0.72379	0.74665
	0.5	0.80051	0.81831
	0.4	0.86169	0.87527
	0.3	0.91067	0.92064
	0.2	0.94949	0.95627
	0.1	0.97927	0.98309
20	0.9	0.33442	0.35913
	0.8	0.51449	0.53360
	0.7	0.64016	0.65503
	0.6	0.73504	0.74665
	0.5	0.80931	0.81831
	0.4	0.86842	0.87527
	0.3	0.91561	0.92064
	0.2	0.95283	0.95627
	0.1	0.98111	0.98309
30	0.9	0.34216	0.35913
	0.8	0.52069	0.53360
	0.7	0.64504	0.65503
	0.6	0.73887	0.74665
	0.5	0.81229	0.81831
	0.4	0.87069	0.87527
	0.3	0.91727	0.92064
	0.2	0.95397	0.95627
	0.1	0.98175	0.98309

Application: Hypothesis Testing

Consider the hypothesis testing scenarios:

$$H_0 : \mu_1 = \mu_2 \quad vs. \quad H_1 : \mu_1 > \mu_2$$
$$H_0 : \mu_1 = \mu_2 \quad vs. \quad H_1 : \mu_1 \neq \mu_2$$

- using doubly-type-II-censored samples
- from two independent location-scale populations Π_i
- with known scale parameters σ_i, $i = 1, 2$
- asymptotic significance level of α
- to have power β^*
- to detect a difference Δ between μ_1 and μ_2

Associated Notations

Let $\xi_t = \Phi^{-1}(1-t)$, with Φ = standard normal c.d.f.

For a fixed α and β^*, define $c_H = c_H(\alpha, \beta^*)$ as:

for single-sided hypothesis

$$1 - \Phi\left(\xi_\alpha - c_H\right) = \beta^*,$$

for double-sided hypothesis

$$1 - \Phi\left(\xi_{\alpha/2} - c_H\right) - \Phi\left(-\xi_{\alpha/2} - c_H\right) = \beta^*.$$

For $i = 1, 2$

n_i^H = total number of observations from population Π_i

with $n_i^H(1 - p_{i,1} - p_{i,2})$ uncensored observations

Procedure $\mathcal{R}.4$

From observations $\vec{\mathbf{Y}}_i$, *estimate* $\widehat{\mu}_i$, $i = 1, 2$.
Then calculate the test-statistic

$$T(\vec{\mathbf{Y}}_1, \vec{\mathbf{Y}}_2) = \frac{\widehat{\mu}_1 - \widehat{\mu}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1^H\, B(p_{1,1}, p_{1,2})} + \dfrac{\sigma_2^2}{n_2^H\, B(p_{2,1}, p_{2,2})}}}. \qquad (4)$$

For single-sided hypothesis reject H_0 *if* $T > \xi_\alpha$
For double-sided hypothesis reject H_0 *if* $|T| > \xi_{\alpha/2}$.

Theorem 5

If n_i^H, $i = 1, 2$ *are taken to be*

$$n_i^H \geq \frac{2\, c_H^2 \sigma_i^2}{\Delta^2 B(p_{i,1}, p_{i,2})}, \qquad (5)$$

then Procedure 4 asymptotically provides power $\geq\ \beta^*$.

Specific Location-Scale Distributions

Normal Distribution

$$B_0(t) = -\frac{d^2}{dt^2} \log \phi(t) = 1.$$

$$\Rightarrow \quad B_N(p_1, p_2) = 1 - p_1 - p_2 + \phi(q_1) \left[\frac{\phi(q_1)}{p_1} + q_1 \right] + \phi(q_2) \left[\frac{\phi(q_2)}{p_2} - q_2 \right].$$

Smallest Extreme Value Distribution
Let $G(t) = 1 - \exp(-\exp(t))$ and $g(t) = \exp(t - \exp(t))$.
$B_0(t)$ simplifies to $B_0(t) = \exp(t)$
and $\int B_0(t) g(t) dt = G(t) - g(t)$.

$$\Rightarrow \quad B_{SEV}(p_1, p_2) = 1 - p_1 - p_2 + \frac{1 - p_1}{p_1} (\ln(1 - p_1))^2.$$

Specific Location-Scale Distributions - cont'd

Largest Extreme Value Distribution

Let $G(t) = \exp(-\exp(-t))$ and $g(t) = \exp(-t - \exp(-t))$.
$\Rightarrow B_0(t) = \exp(-t)$ and $\int B_0(t)g(t)dt = G(t) + g(t)$

$$\Rightarrow \quad B_{LEV}(p_1, p_2) = 1 - p_1 - p_2 + \frac{1-p_2}{p_2}(\ln(1-p_2))^2.$$

Logistic Distribution

Let $G(t) = \dfrac{\exp(t)}{1+\exp(t)}$ and $g(t) = \dfrac{\exp(t)}{(1+\exp(t))^2}$.
$B_0(t) = 2g(t)$.

$$\Rightarrow \quad B_{LOGIS}(p_1, p_2) = \frac{2p_2^3}{3} - p_2^2 - \frac{2(1-p_1)^3}{3} + (1-p_1)^2 + \frac{p_1^2(1-p_1)^2}{p_1} - p_1(1-p_1)(1-2p_1) + \frac{p_2^2(1-p_2)^2}{p_2} - p_2(1-p_2)(2p_2-1).$$

Associated Log-Location-Scale Distributions

- $Y = \ln X \sim N(\mu, \sigma^2) \Rightarrow X \sim \Lambda(\mu, \sigma^2)$
- $Y = \ln X \sim SEV(\mu, \sigma) \Rightarrow X \sim \text{Weibull}(\exp(\mu), 1/\sigma)$
  - $EX^m = \exp(m\mu)\Gamma(1 + m\sigma)$
  - $\tau_\alpha = \exp(\mu + G^{-1}(\alpha)\sigma)$
- $Y = \ln X \sim \text{Logistic}(\mu, \sigma) \Rightarrow X \sim \text{Log} - \text{Logistic}(\mu, \sigma^2)$
  - $EX^m = \exp(m\mu)\Gamma(1 + m\sigma)\Gamma(1 - m\sigma)$
  - $\text{median}(X) = \exp(\mu)$,
  - $\tau_\alpha = \exp(\mu + G^{-1}(\alpha)\sigma)$

Concluding Remarks

Related Work:

- Properties of $B(p_1, p_2)$ [John and Chen(2008)]
- Selection of the Best
  - from (Log-)Location-Scale
  - complete samples or Type-II Censored
  - with σ_i^2 Known [John and Chen(2008)]
- Procedure for Type-II Censored with σ_i^2 Unknown [John and Chen(2006b)]

Some Directions for Future Research:

- Optimal Initial Sample Sizes
- Other types of Censoring (Type-I, Interval, Random)
- Selection from Accelerated Life Tests

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